1.

We are going to appeal to the so-called Zeckendorf theorem:

Theorem

Every positive integer can be uniquely expressed as the sum of nonconsecutive Fibonacci numbers.

The idea is to take the following so-called *Wythoff array*:

- $1, 2, 3, 5, \ldots$
- $1+3, 2+5, 3+8, \ldots$
- $1+5, 2+8, 3+13, \ldots$
- $1+8, 2+13, 3+21, \ldots$
- $1+3+8, 2+5+13, 3+8+21, \ldots$
- $\ldots$

We write the details below.

Let $\{F_i\}$ denote the Fibonacci numbers with $F_1 = 1, F_2 = 2$. Say $n = \overline{a_k \cdots a_1}_{\text{Fib}}$ with $a_k = 1$ is a Fibonacci base representation of n if a_i is 0 or 1,

$$n = \sum_{i=1}^{k} a_i F_i$$

and a_i, a_{i+1} are not both 1 for any i. Equivalently, it is a representation of n as a sum of nonconsecutive Fibonacci numbers.

We begin by outlining a proof of Zeckendorf's theorem, which implies the representation above is unique. Note that if F_k is the greatest Fibonacci number at most n, then

$$n - F_k < F_{k+1} - F_k = F_{k-1}.$$

In particular, repeatedly subtracting off the largest F_k from n will produce one such representation with no two consecutive Fibonacci numbers. On the other hand, this F_k must be used, as

$$n \geq F_k > F_{k-1} + F_{k-3} + F_{k-5} \cdots$$

This shows, by a simple inductive argument, that such a representation exists and unique.

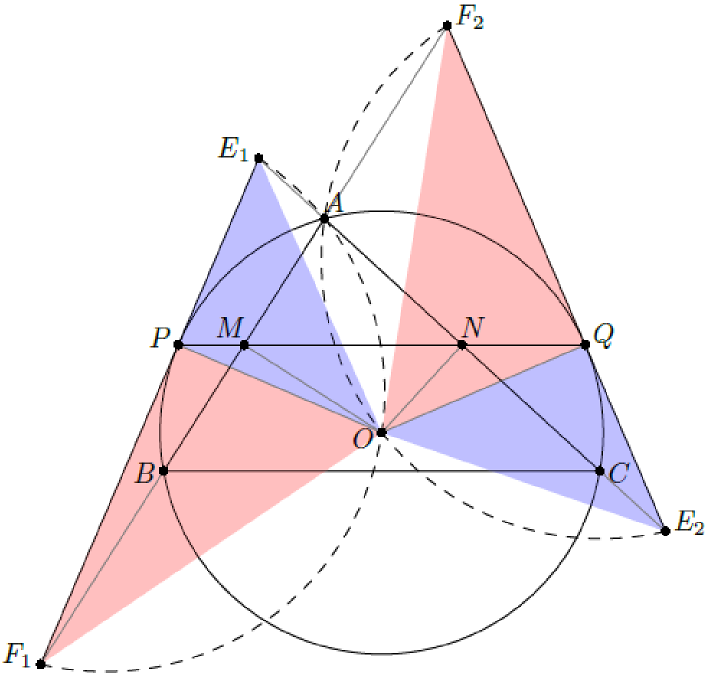
Now for each $\overline{a_k \cdots a_1}_{\text{Fib}}$ with $a_1 = 1$, consider the sequence

$$\overline{a_k \cdots a_1}_{\text{Fib}}, \overline{a_k \cdots a_1 0}_{\text{Fib}}, \overline{a_k \cdots a_1 00}_{\text{Fib}}, \ldots$$

These sequences are Fibonacci-type by definition, and partition the positive integers since each positive integer has exactly one Fibonacci base representation.

2.

Let O be the center of ω, and let $M = \overline{PQ} \cap \overline{AB}$ and $N = \overline{PQ} \cap \overline{AC}$ be the midpoints of $\overline{AB}$ and $\overline{AC}$ respectively. Refer to the diagram below.



The main idea is to prove two key claims involving O, which imply the result:

(i) quadrilaterals AOE_1F_1 and AOE_2F_2 are cyclic (giving the radical axis is $\overline{AO}$),

(ii) $\triangle OE_1F_1 \cong \triangle OE_2F_2$ (giving the congruence of the circles).

We first note that (i) and (ii) are equivalent. Indeed, because $OP = OQ$, (ii) is equivalent to just the similarity $\triangle OE_1F_1 \sim \triangle OE_2F_2$, and then by the spiral similarity lemma (or even just angle chasing) we have (i) $\iff$ (ii).

We now present five proofs, two of (i) and three of (ii). Thus, we are essentially presenting five different solutions.

Proof of (i) by angle chasing Note that

$$\measuredangle F_2E_2O = \measuredangle QE_2O = \measuredangle QNO = \measuredangle MNO = \measuredangle MAO = \measuredangle F_2AO$$

and hence E_2OAF_2 is cyclic. Similarly, E_1OAF_1 is cyclic.

Proof of (i) by Simson lines Since P, M, N are collinear, we see that $\overline{PMN}$ is the Simson line of O with respect to $\triangle AE_1F_1$.

Proof of (ii) by butterfly theorem By BUTTERFLY THEOREM on the three chords $\overline{AC}$, $\overline{PQ}$, $\overline{PQ}$, it follows that $E_1N = NE_2$. Thus

$$E_1P = \sqrt{E_1A \cdot E_1C} = \sqrt{E_2A \cdot E_2C} = E_2P.$$

But also $OP = OQ$ and hence $\triangle OPE_1 \cong \triangle OQE_2$. Similarly for the other pair.

Proof of (ii) by projective geometry Let $T = \overline{PP} \cap \overline{QQ}$. Let S be on $\overline{PQ}$ with $\overline{ST} \parallel \overline{AC}$; then $\overline{TS} \perp \overline{ON}$, and it follows $\overline{ST}$ is the polar of N (it passes through T by La Hire).

Now,

$$-1 = (PQ; NT) \stackrel{T}{=} (E_1E_2; N\infty)$$

with $\infty = \overline{AC} \cap \overline{ST}$ the point at infinity. Hence $E_1N = NE_2$ and we can proceed as in the previous solution.

Proof of (ii) by complex numbers We will give using complex numbers on $\triangle ABC$ a proof that $|E_1P| = |E_2Q|$.

We place $APBCQ$ on the unit circle. Since $\overline{PQ} \parallel \overline{BC}$, we have $pq = bc$. Also, the midpoint of $\overline{AB}$ lies on $\overline{PQ}$, so

$$\begin{aligned} p + q &= \frac{a+b}{2} + \overline{\left(\frac{a+b}{2}\right)} \cdot pq \\ &= \frac{a+b}{2} + \frac{a+b}{2ab} \cdot bc \\ &= \frac{a(a+b)}{2a} + \frac{c(a+b)}{2a} \\ &= \frac{(a+b)(a+c)}{2a}. \end{aligned}$$

Let $p_1, p_2, p_3, \cdots$, be the sequence of primes.
Given $k > 0$, consider a square matrix with order k whose entries in the first row consists of first k primes, those in the the second row consists of the next k primes, and so on. Let m_i be the product of the primes in the n^{th} row and let M_i be the product of the primes in the i^{th} column. Then, the numbers m_i are relatively prime in pairs, as are the M_i.

Next, consider the set of congruences.

$$\begin{aligned} x &\equiv -1 \mod m_1 \\ x &\equiv -2 \mod m_2 \\ &\vdots \\ x &\equiv -k \mod m_k \end{aligned}$$

This system has a solution a which is unique mod $m_1 m_2 \cdots m_k$.
Similarly, the system

$$\begin{aligned} y &\equiv -1 \mod M_1 \\ y &\equiv -2 \mod M_2 \\ &\vdots \\ x &\equiv -k \mod M_k \end{aligned}$$

has a solution b which is unique mod $M_1 \cdots M_k = m_1 \cdots m_k$.

Now consider the square with the opposite vertices (a, b) and $(a+k, b+k)$. Any lattice point inside this square has the form $(a+r, b+s)$, where $0 < r < k, 0 < s < k$, and those with $r = k$ or $s = k$ lie on the boundary of the square. Now, we show that no such point is visible from the origin.
In fact, $a \equiv -r \pmod{m_r}$ and $b \equiv -s \pmod{M_s}$. So, the prime in the intersection of row r and column s divides both $a+r$ and $b+s$. Hence, $a+r$ and $b+s$ are not are not relatively prime and hence the lattice point $(a+k, b+k)$ is not visible from the origin.

4.

First solution: We will prove that the maximum value of n is 11. Figure 105 describes an arrangement of 12 dominoes such that no additional domino can be placed on the board. Therefore, $n \leq 11$.

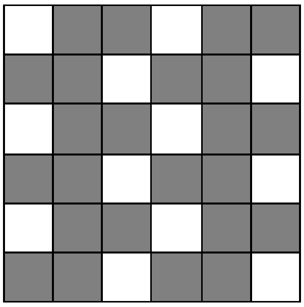


Figure 105

Let us show that for any arrangement of 11 dominoes on the board one can add one more domino. Arguing by contradiction, let us assume that there is a way of placing 11 dominoes on the board so that no more dominoes can be added. In this case there are $36 - 22 = 14$ squares not covered by dominoes.

Denote by S_1 the upper 5×6 subboard, by S_2 the lower 1×6 subboard, and by S_3 the lower 5×6 subboard of the given chessboard as shown in Figure 106.

Because we cannot place another domino on the board, at least one of any two neighboring squares is covered by a domino. Hence there are at least three squares in S_2 that are covered by dominoes, and so in S_2 there are at most three uncovered squares. If A denotes the set of uncovered squares in S_1, then $|A| \geq 14 - 3 = 11$.

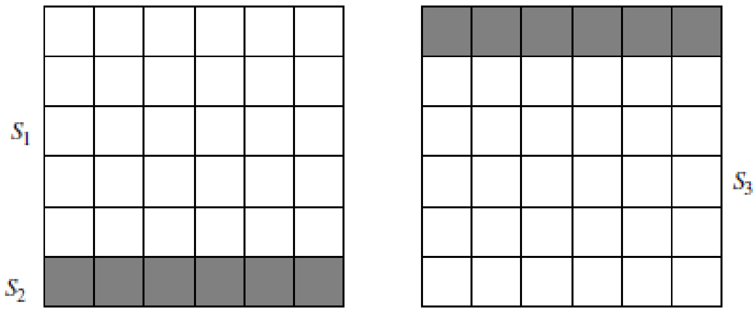


Figure 106

Let us also denote by B the set of dominoes that lie completely in S_3. We will construct a one-to-one map $f : A \to B$. First, note that directly below each square s in S_1 there is a square t of the chessboard (see Figure 107). If s is in A, then t must be covered by a domino d in B, since otherwise we could place a domino over s and t. We define

$f(s) = d$. If f were not one-to-one, that is, if $f(s_1) = f(s_2) = d$, for some $s_1, s_2 \in A$, then d would cover squares directly below s_1 and s_2 as described in Figure 107. Then s_1 and s_2 must be neighbors, so a new domino can be placed to cover them. We conclude that f is one-to-one, and hence $|A| \leq |B|$. It follows that $|B| \geq 11$. But there are only 11 dominoes, so $|B| = 11$. This means that all 11 dominoes lie completely in S_3 and the top row is not covered by any dominoes! We could then put three more dominoes there, contradicting our assumption on the maximality of the arrangement. Hence the assumption was wrong; one can always add a domino to an arrangement of 11 dominoes. The answer to the problem is therefore $n = 11$.

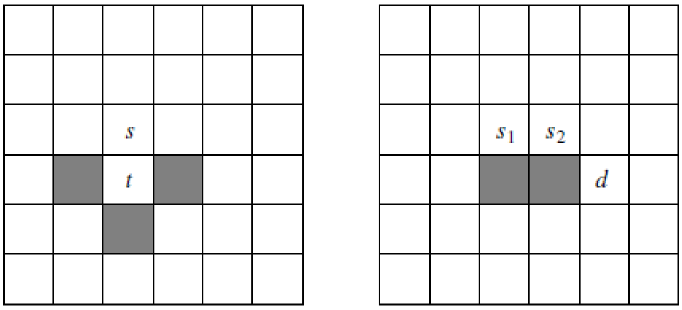


Figure 107

Second solution: Suppose we have an example with k dominoes to which no more can be added. Let X be the number of pairs of an uncovered square and a domino that covers an adjacent square. Let $m = 36 - 2k$ be the number of uncovered squares, let m_∂ be the number of uncovered squares that touch the boundary (including corner squares), and m_c the number of uncovered corner squares. Since any neighbor of an uncovered square must be covered by some domino, we have $X = 4m - m_\partial - m_c$. Similarly, let k_∂ be the number of dominoes that touch the boundary and k_c the number of dominoes that contain a corner square. A domino in the center of the board can have at most four unoccupied neighbors, for otherwise, we could place a new domino adjacent to it. Similarly, a domino that touches the boundary can have at most three unoccupied neighbors, and a domino that contains a corner square can have at most two unoccupied neighbors. Hence $X \leq 4k - k_\partial - k_c$. Also, note that $k_\partial \geq m_\partial$, since as we go around the boundary we can never encounter two unoccupied squares in a row, and $m_c + k_c \leq 4$, since there are only four corners. Thus $4m - m_\partial - m_c = X \leq 4k - k_\partial - k_c$ gives $4m - 4 \leq 4k$; hence $35 - 2k \leq k$ and $3k \geq 35$. Thus k must be at least 12. This argument also shows that on an $n \times n$ board, $3k^2 \geq n^2 - 1$.

5.

Let us make the convention that the letter p always denotes a prime number. Consider the set $A(n)$ consisting of those positive integers that can be factored into primes that do not exceed n. Then

$$\prod_{p\le n}\left(1+\frac{1}{p}+\frac{1}{p^2}+\cdots\right)=\sum_{m\in A(n)}\frac{1}{m}.$$

This sum includes $\sum_{m=1}^{n}\frac{1}{m}$, which is known to exceed $\ln n$. Thus, after summing the geometric series, we obtain

$$\prod_{p\le n}\left(1-\frac{1}{p}\right)^{-1}>\ln n.$$

For the factors of the product we use the estimate

$$e^{t+t^2}\ge(1-t)^{-1},\quad \text{for } 0\le t\le\frac{1}{2}.$$

To prove this estimate, rewrite it as $f(t)\ge 1$, where $f(t)=(1-t)e^{t+t^2}$. Because $f'(t)=t(1-2t)e^{t+t^2}\ge 0$ on $[0,\frac{1}{2}]$, f is increasing; thus $f(t)\ge f(0)=1$.

Returning to the problem, we have

$$\prod_{p\le n}\exp\left(\frac{1}{p}+\frac{1}{p^2}\right)\ge\prod_{p\le n}\left(1-\frac{1}{p}\right)^{-1}>\ln n.$$

Therefore,

$$\sum_{p\le n}\frac{1}{p}+\sum_{p\le n}\frac{1}{p^2}>\ln\ln n.$$

But

$$\sum_{p\le n}\frac{1}{p^2}<\sum_{n=2}^{\infty}\frac{1}{k^2}=\frac{\pi^2}{6}-1<1.$$

Hence

$$\sum_{p\le n}\frac{1}{p}\ge\ln\ln n-1,$$

as desired.